

Calculations with DLCQ¹

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Abstract

The method of discrete light-cone quantization (DLCQ) and useful refinements are summarized. Applications to various field theories are reviewed.

1 Introduction

Nonperturbative solutions are of critical importance for the full understanding and application of quantum field theories. One method to obtain such solutions is discrete light-cone quantization (DLCQ) [1,2]. A brief summary of this method, its applications, and some refinements are given below. The summary is much too brief to be tutorial; an expanded discussion can be found in Ref. [2]. Also due to brevity, other promising light-cone methods, such as the transverse lattice [3] and similarity transformations [4], are not discussed.

DLCQ builds from the imposition of periodic or antiperiodic boundary conditions on a light-cone box $-L < x^- < L$, $-L_\perp < x, y < L_\perp$. Usually couplings dictate that periodic boundary conditions be used for bosons. Either form can be chosen for fermions, with antiperiodic preferred in order to avoid zero modes. In momentum space there is then a discrete grid: $p^+ \rightarrow n\pi/L$, $\mathbf{p}_\perp \rightarrow (n_x\pi/L_\perp, n_y\pi/L_\perp)$, with n even for periodic boundary conditions and odd for antiperiodic. The limit $L \rightarrow \infty$ is exchanged for a limit in terms of the integer *harmonic resolution* $K \equiv \frac{L}{\pi}P^+$ [1]. It sets the resolution for the

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longitudinal momentum fractions $x = p^+/P^+ \rightarrow n/K$. The light-cone Hamiltonian $H_{\text{LC}} \equiv P^+P^-$ is independent of L . Because p^+ is positive, the harmonic resolution limits the range of n to be no more than K and the number of constituents to be no more than roughly $K/2$. There is no corresponding limit for the transverse components, and an additional constraint must be imposed. One frequently used is the invariant mass cutoff $m_i^2 + p_{\perp i}^2 \leq x_i \Lambda^2$. The transverse integers then lie within some finite range, $-N_{\perp} \leq n_x, n_y \leq N_{\perp}$.

The discretization, combined with Fock-state expansions for the eigenstates of H_{LC} , produces a finite matrix approximation to the infinite set of coupled integral equations for the Fock-state wave functions. The integrals are replaced by trapezoidal approximations such as

$$\int_0^1 dx \int d^2 p_{\perp} f(x, \mathbf{p}_{\perp}) \simeq \frac{2}{K} \left(\frac{\pi}{L_{\perp}} \right)^2 \sum_n \sum_{n_x, n_y = -N_{\perp}}^{N_{\perp}} f(n/K, \mathbf{n}_{\perp} \pi/L_{\perp}), \quad (1)$$

with nominal errors of order $1/K^2$ for nonsingular kernels. Additional truncation in particle number is frequently applied, to keep the matrix representation small.

For large matrices the diagonalization problem is best attacked with the Lanczos algorithm [5] which requires of the matrix only matrix-vector multiplications. This takes good advantage of the usually sparse structure of the matrix by allowing optimal storage or even computation of matrix elements as needed. An alternative to explicit diagonalization is reduction to an effective Hamiltonian in a single Fock sector [6].

2 Brief review of calculations

There have been a number of applications of this method. The earliest is a nonrelativistic test [7], but all the others are applications to field theories in two or more dimensions. The first field-theoretic application was by Pauli and Brodsky to two-dimensional Yukawa theory [1]. Other two-dimensional applications include ϕ^n [8], QED [9] (including coherent states [10] and QED at finite temperature [11]), QCD [12], adjoint matter (including tube models and collinear models) [13], and the following models: Wick–Cutkosky [14], sine–Gordon [15], Gross–Neveu [16], and Abelian Higgs [17]. The behavior of the Pauli–Jordan function $\Delta(x) = -i \int \frac{d^2 k}{2\pi} \delta(k^2 - m^2) \epsilon(k^0) e^{-ik \cdot x}$, which should be zero for $x^2 < 0$, has been checked as a test of microcausality [18]. Four-dimensional applications include QED [19] (including coherent states [20]), positronium [21], a perturbative calculation of the electron’s anomalous moment [22], QCD [23], the Wick–Cutkosky model [24], a dressed fermion model

[25] (including a calculation of the F_1 form factor [26]), and Yukawa theory in a single-fermion truncation [27]. The dressed fermion calculation [25] is the largest to date, having used basis sizes on the order of 10 million.

An important, recurring issue is that of zero modes [2]. These take several forms: ghost fields required for a consistent quantization, constrained fields associated with the periodic boundary conditions, and dynamical zero modes of periodic gauge fields. The zero modes are intimately tied with the representation of what in equal-time quantization is the structure of the vacuum. Even if this structure is trivial, zero modes represent corrections of order $1/K$ to DLCQ calculations that ignore them. These $1/K$ effects can be introduced through effective interactions [24,28].

To quickly see the potential importance of zero modes, consider the Wick–Cutkosky model in two dimensions, for which the interaction Lagrangian is $-g\phi|\chi|^2$. We impose periodic boundary conditions for ϕ and extract the zero mode field $\phi_0 = \int_{-L}^L \phi dx^- / 2L$. Antiperiodic boundary conditions are used for the complex scalar χ . By integrating the equation of motion, we obtain the constraint $\mu^2 \phi_0 = -g \frac{1}{2L} \int_{-L}^L dx^- |\chi|^2$. This implies that the interaction term of the Hamiltonian will include a term of the form $-\frac{g^2}{\mu^2} \frac{1}{2L} \left(\int_{-L}^L dx^- |\chi|^2 \right)^2$, which is unbounded from below, for any g , in the limit of infinite resolution. DLCQ calculations without zero modes can see this unbounded spectrum [14], but with some difficulty.

Recent work on zero modes includes chiral symmetry breaking [29], symmetry breaking in ϕ^4 [30] (including a treatment that does not require zero modes [31]), an additional effective interaction in the massive Schwinger model [32], and the chiral Yukawa model [33]. For earlier work, see the review in Ref. [2].

3 Refinements of DLCQ

Since the original formulation of DLCQ [1] several refinements and variants of the method have appeared. Those described briefly here are corrections for end-point behavior [34], the use of unequal integration weights [25], an indefinite-metric Lanczos algorithm [27], supersymmetric DLCQ [35,36], and treatment of scattering amplitudes [37].

When a constituent mass is small, there can be significant end-point corrections to DLCQ wave functions. The introduction of an effective interaction to the DLCQ Hamiltonian can compensate for this [34]. The effective interaction is constructed to include the leading end-point behavior exactly.

Alternative integration schemes introduce unequal weights at grid points near

the boundaries [38,25] to compensate for the DLCQ grid being incommensurate with the invariant-mass cutoff. The weights can be obtained by iterating one-dimensional integration rules and by picking the one-dimensional rules to satisfy chosen constraints, such as exact integration of linear forms. Additional improvement can be obtained by taking into account the cylindrical symmetry of both the integration domain and the invariant mass constraint for two-body wave functions [25]. The transverse integral is written in polar coordinates, and the radial integral is approximated by a discrete sum over the circles that intersect the points of the square grid.

The recent calculations of the dressed fermion model [25] and Yukawa theory [27] introduce negatively normed Pauli–Villars particles as regulators [39]. The DLCQ matrix representation is then no longer Hermitian. Although there exists a Lanczos diagonalization technique for general matrices, an efficient special form has been developed for this indefinite-metric situation [27]. It produces a tridiagonal representation T which is real and self-adjoint with respect to an induced indefinite metric in the Lanczos basis $\{\vec{q}_k\}$. One can solve $T\vec{c}_i = \lambda_i\vec{c}_i$ for eigenvalues and right eigenvectors and have $H_{\text{LC}}\vec{\phi}_i \simeq \lambda_i\vec{\phi}_i$, with $\vec{\phi}_i = \sum_k (c_i)_k \vec{q}_k$.

Efficient application of DLCQ to supersymmetric theories requires a variant, known as supersymmetric DLCQ (SDLCQ) [35]. It is based on the observation [36] that discretization of the supercharge Q^- and computation of the Hamiltonian P^- from the superalgebra relation $P^- = \frac{1}{2\sqrt{2}} \{Q^-, Q^-\}$ yield a discrete Hamiltonian which is explicitly supersymmetric. The ordinary DLCQ Hamiltonian is not supersymmetric. The two agree only in the limit of infinite resolution. It is expected that zero modes decouple in these theories [40], another advantage of supersymmetry.

An extension of DLCQ to include the calculation of scattering amplitudes has been constructed [37,41]. This generalizes earlier work on the special case of e^+e^- annihilation into hadrons [42]. The invariant amplitude \mathcal{M}_{fi} is obtained from the light-cone T matrix, which is built from individual composite-particle eigenstates and related operators, extending a formulation by Wick [43].

4 Summary

The future of DLCQ holds many exciting prospects, some of which can already be listed. The use of Pauli–Villars regularization can be extended to full Yukawa theory, QED, and perhaps QCD in the form given by Paston *et al.* [44]. Cross sections can be calculated. Symmetry breaking and vacuum structure can be better understood. For those working in string theory, higher resolution SDLCQ calculations in relevant theories will be of considerable interest.

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